# Realizability of Planar Straight-Line Graphs as Straight Skeletons

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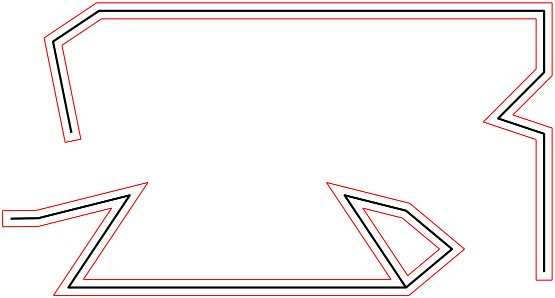
- [Aichholzer and Aurenhammer, 1998]: straight skeleton $\mathcal{S}(G)$ of a PSLG $G$



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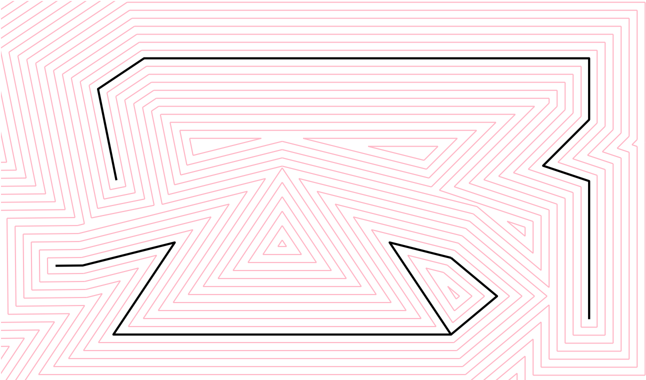

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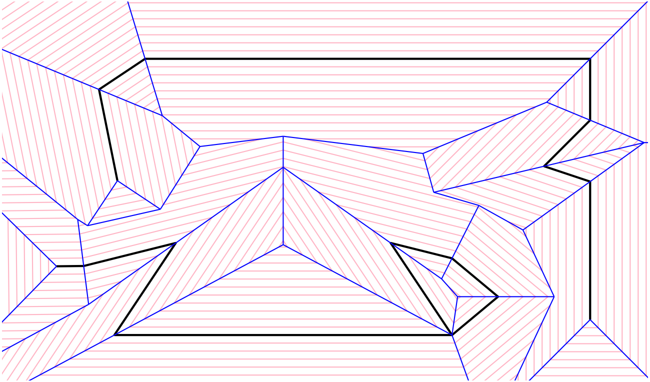
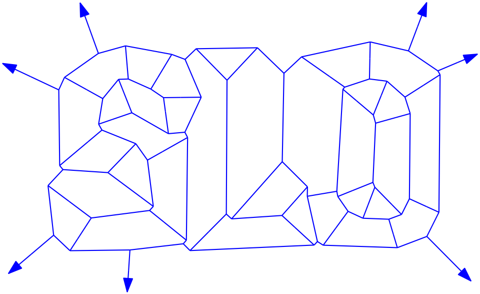


# Problem statement

PSLG$^\infty$: edges may be straight-line segments or rays.

## Problem (GMP-SS)

Given a PSLG$^\infty$ $G$, can we find a PSLG $H$ such that $\mathcal{S}(H) = G$?



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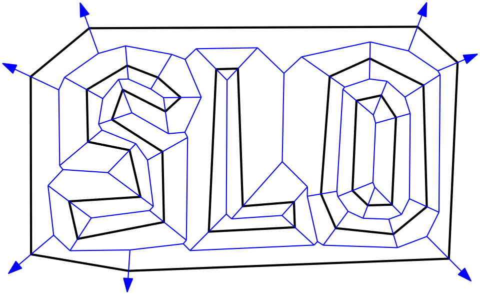

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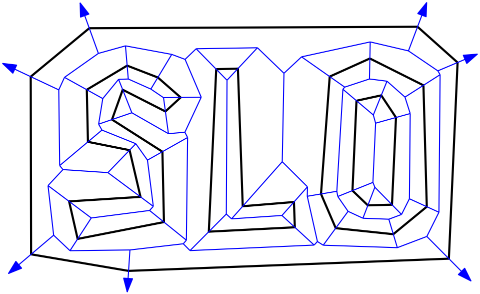


## Problem [Aichholzer et al., 1995]

Give necessary and sufficient conditions for $G$ to be the straight skeleton of $H$.

# Prior work

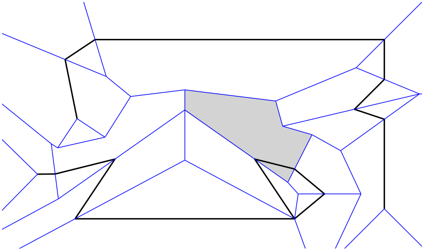
[Aichholzer et al., 2012]:

- Any abstract tree $T$ can be realized as $\mathcal{S}(P)$ (or $\mathcal{V}(P)$) of a convex polygon.
- Realizability of phylogenetic trees $T$ as $\mathcal{S}(P)$ of a polygon $P$.

# Characterization: basic facts

## Facts

- Every edge of $\mathcal{S}(H)$ is on the bisector of two edges of $H$.
- Every face of $\mathcal{S}(H)$ contains exactly one segment of $H$, except for faces generated by degree-one vertices of $H$.
- Every edge of $H$ begins and ends at an edge of $\mathcal{S}(H)$.
- If a vertex of $\mathcal{S}(H)$ has degree two then it coincides with a degree-one vertex of $H$. All other vertices have degree three or higher.



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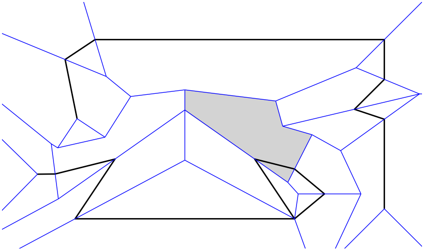
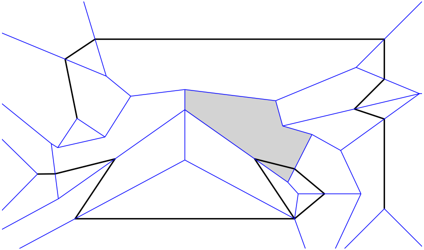


**Temporary assumption:** $G$ has no degree-2 vertices.

# Characterization: inside-condition

Let $G$ be the putative straight skeleton and $F$ the set of faces of $G$.



A solution to GMP-SS can be denoted as a mapping $\lambda \colon F \to \mathcal{L}$, where $\mathcal{L}$ is the set of lines.

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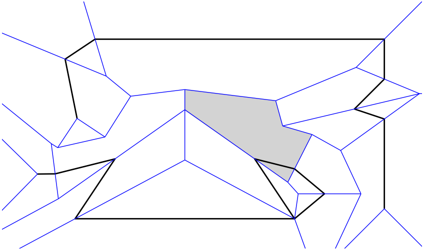


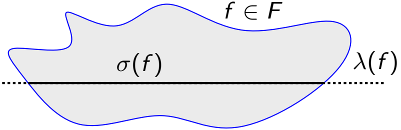
A solution to GMP-SS can be denoted as a mapping $\lambda\colon F \to \mathcal{L}$, where $\mathcal{L}$ is the set of lines.

**Definition (Inside-condition)**

$\lambda$ fulfills the inside-condition if $\sigma(f) := \lambda(f) \cap f$ is a single line segment for all $f \in F$.

# Characterization: inside-condition

We construct $H$ as the graph whose edges are $\sigma(f)$, with $f \in F$.



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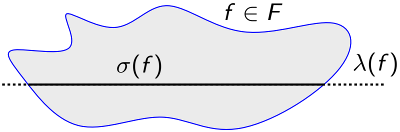



For a $G$ and $\lambda$ we denote by $G^* := G \cup H$ and by $F^*$ the faces of $G^*$.

- Every face of $G$ contains two faces of $G^*$.
- We reuse $\lambda(f)$ and $\sigma(f)$ for faces of $G^*$ accordingly.

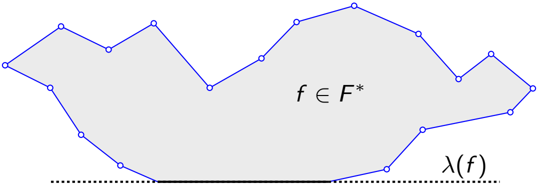
# Characterization: sweeping-condition

## Definition (Sweeping-condition)

A face $f$ of $G^*$ fulfills the sweeping-condition if

1. $f$ is **monotone** w.r.t. $\lambda(f)$ and
2. at the lower chain, the **distance to $\lambda(f)$ is increasing**, when moving away from $\sigma(f)$.

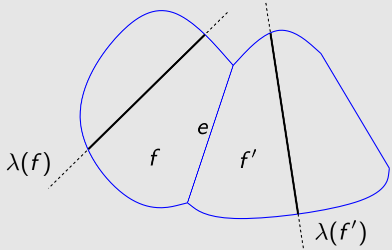
$\lambda$ fulfills the sweeping-condition if all faces of $G^*$ fulfill it.



# Characterization: bisector-condition

## Definition (Bisector-condition)

The edge $e = f \cap f'$ fulfills the bisector-condition if $e$ lies on the bisector of $\lambda(f)$ and $\lambda(f')$.




$\lambda$ fulfills the bisector-condition if all edges of $G$ fulfill the bisector-condition.

# Characterization

## Lemma

*If $\lambda$ solves GMP-SS then $\lambda$ fulfills the inside-, sweeping-, and bisector-condition.*

*Proof.* Inside- and bisector-condition: by definition of straight skeletons.
Sweeping-condition:

- Monotonicity by [Aichholzer et al., 1995].
- Lower chain is even convex by [Huber, 2012].

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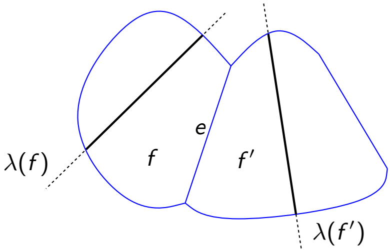
**Theorem**

*If $\lambda$ fulfills the inside-, sweeping-, and bisector-condition then $\lambda$ solves GMP-SS.*

# Recognizing straight skeletons

**Key method:** We successively reflect lines $\lambda(f)$ at edges of $f$.

- Assume we know a suitable $\lambda(f)$ for one face $f$.
- Bisector-condition: we know $\lambda(f')$ for a neighboring face $f'$, too.
- Going along a spanning tree of the dual of $G$, we know $\lambda(f')$ for all $f' \in F$.



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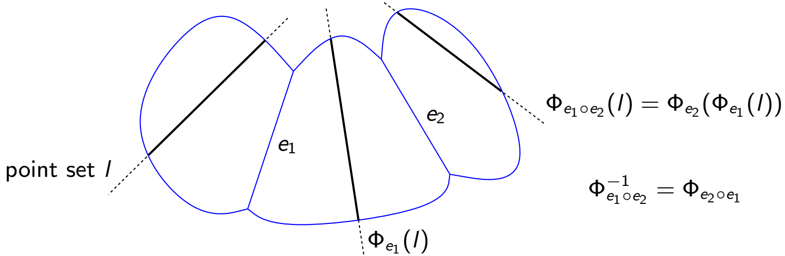
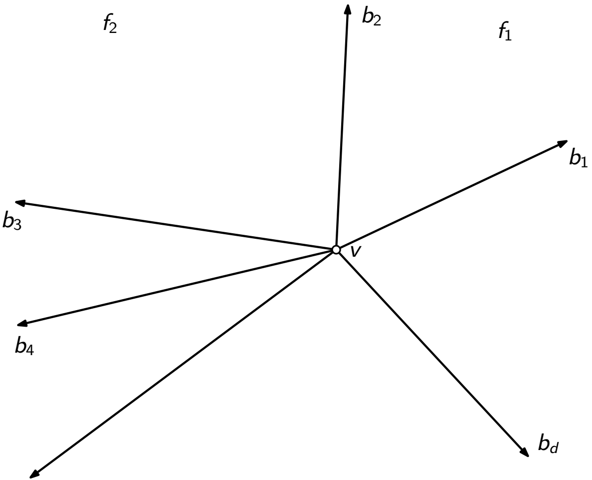


# Recognizing straight skeletons: star graphs

- "Local view" at a vertex $v$ of $G$ with incident ray-edges $b_1, \ldots, b_d$.
  - Find $\lambda$ that fulfills inside-, (sweeping-), and bisector-condition.



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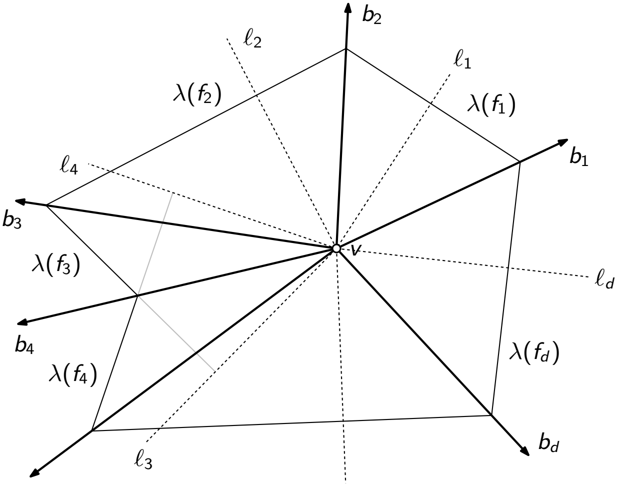

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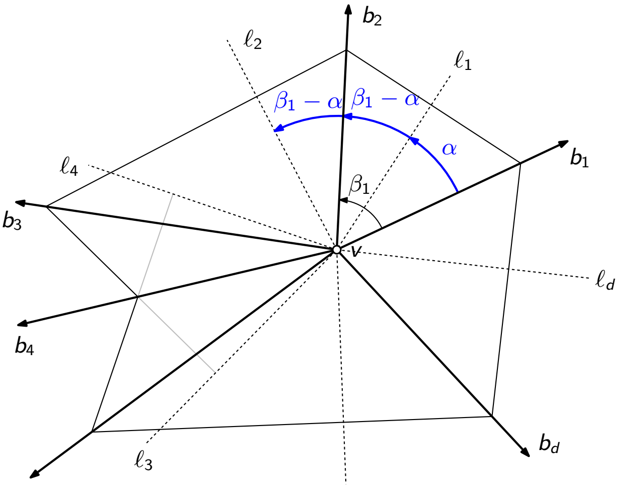

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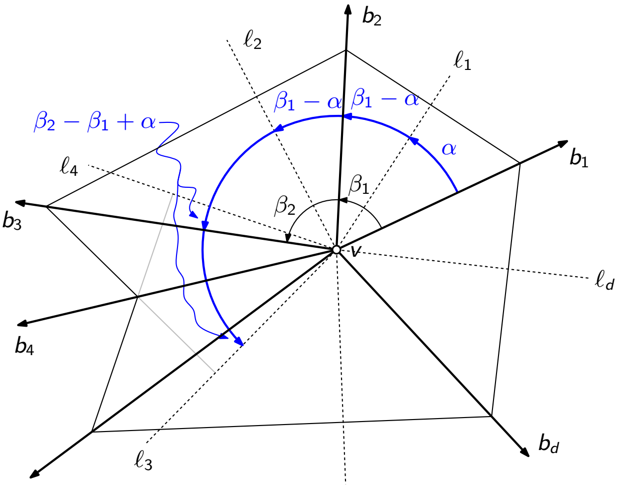

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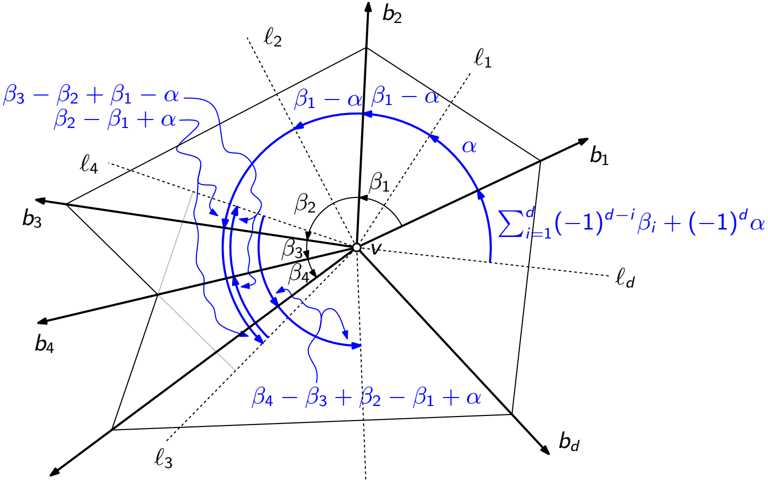

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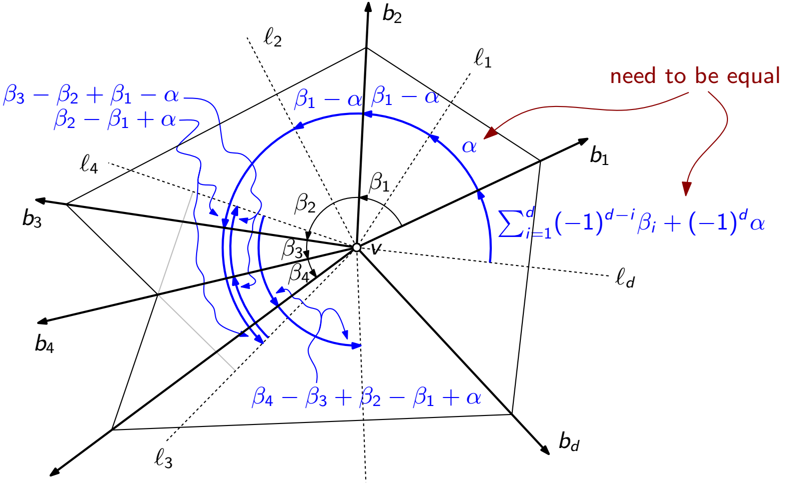

# Recognizing straight skeletons: star graphs

We get $\alpha = \sum_{i=1}^{d} (-1)^{d-i} \beta_i + (-1)^d \alpha$ and therefore

$$\frac{1}{2} \sum_{i=1}^{d} (-1)^{d-i} \beta_i = \begin{cases} 0 & \text{if } d \text{ is even,} \\ \alpha & \text{if } d \text{ is odd.} \end{cases} \tag{1}$$

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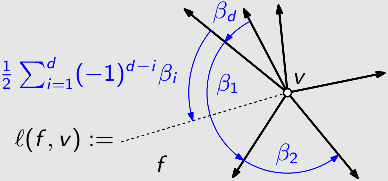
## Definition

The vertex $v$ with even degree $d$ fulfills the **balance-condition** if $\beta_d - \beta_{d-1} + \cdots + \beta_2 - \beta_1 = 0$.

For vertices of odd degree $d$ we define $\ell(f, v)$ as

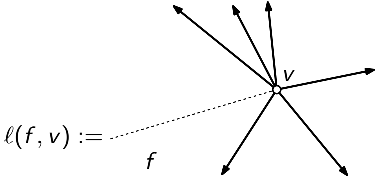


# Recognizing straight skeletons: star graphs

## Lemma

$\Phi_{b_1 \circ \cdots \circ b_d}(l) = l$ *if and only if*

$$\begin{cases} v \text{ fulfills the balance-condition} & \text{if } d \text{ is even} \\ l = \ell(f,v) \ \vee \ l \perp \ell(f,v) \text{ for some } f \in F \text{ with } v \in f & \text{if } d \text{ is odd} \end{cases} \qquad (2)$$



# Recognizing straight skeletons: PSLGs

The previous lemma imposes constraints on $\lambda$ for the vertices of $G$:

$$\ell(f) := \{l \in \mathcal{L} : \; l \cap \operatorname{int} f \neq \emptyset\} \; \cap \bigcap_{\substack{v \text{ is vertex of } f \\ \deg(v) \text{ is odd}}} \{\ell(f,v)\} \cup \ell(f,v)^{\perp}. \qquad (3)$$

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We propagate the per-face constraints to a single face:

- Choose a spanning tree $T$ of the dual of $G$, with a root face $r$.
- Denote by $f \rightsquigarrow^T r$ the sequence of edges in $T$ from $f$ to $r$ and define

$$f^r := \Phi_{f \rightsquigarrow^T r}(f) \tag{4}$$

$$\ell^r(f) := \Phi_{f \rightsquigarrow^T r}(\ell(f)) \tag{5}$$

$$X := \bigcap_{f \in F} \ell^r(f). \tag{6}$$

# Recognizing straight skeletons: PSLGs

**Theorem**

*GMP-SS for $G$ has a solution if and only if*

- *the balance-condition holds for all vertices of even degree and*
- *there is a line $l \in X$ such that for all $f \in F$*
  - *$l \cap f^r$ is a single segment and*
  - *the components of $f^r \setminus l$ fulfill the sweeping-condition.*

*There is a one-to-one correspondence between such lines $l \in X$ and solutions to GMP-SS.*

# Recognizing straight skeletons: PSLGs

Proof sketch:

- Take a suitable $l$ and define $\lambda(f) := \Phi_{r \rightsquigarrow_T f}(l)$.
- To show: $\lambda$ fulfills the inside-, bisector- and sweeping-condition.
  - Inside- and sweeping-condition are fulfilled by assumption.
  - Bisector-condition for (duals of) edges in $T$ as well.

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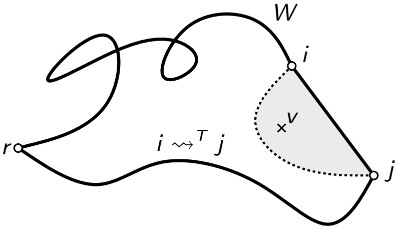
- **Claim:** edges not in $T$ fulfill the bisector-condition as well.
- **Stronger claim:** Let $W$ be any walk in the dual of $G$ from $r$ to $j$. Then $\Phi_W(\lambda(r)) = \lambda(j)$. That is, it does not matter how we choose $T$.



# Reconstructing the input: algorithm

We are given $G$ and want to find a suitable $\lambda$, i.e., a suitable $I \in X$.

- Check that balance-condition holds at every even-degree vertex.
- We compute $T$, all $f^r = \Phi_{f \rightsquigarrow T_r}(f)$ and all $\ell^r(f, v) = \Phi_{f \rightsquigarrow T_r}(\ell(f, r))$ in total linear time.

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- **Case 1:** All vertices have even degree.
  - By the balance-condition all faces are convex.
    - Sweeping-condition is trivial.
  - Using [Edelsbrunner et al., 1989] and [Hershberger, 1989] we find all lines $l$ traversing all int $f^r$ in $O(n \log n)$ time.

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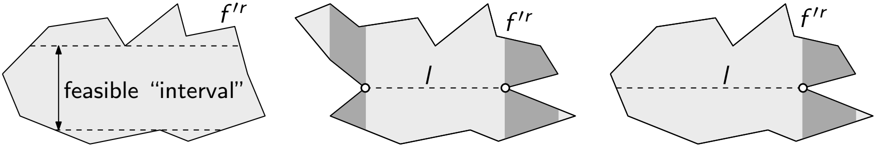
- **Case 2:** At least one vertex $v$ has odd degree.
  - A suitable $l$ has fixed direction: identical or perpendicular to $\ell^r(f, v)$.
  - inside/sweeping condition $\Rightarrow$ restrict all suitable $l$ to an "interval" of parallel lines in $O(n)$ time.



# Reconstructing the input

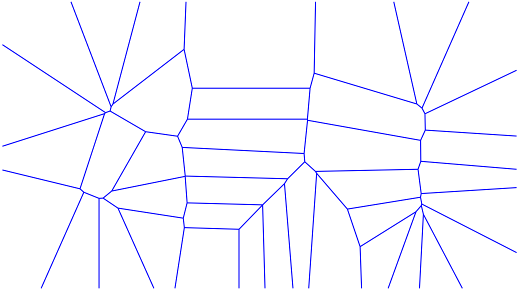
## Theorem

*GMP-SS can be solved and the set of solutions can be found in $O(n \log n)$ time of a PSLG$^\infty$ $G$ with $n$ edges.*

# Voronoi diagrams

## Problem (GMP-VD)

Given a PSLG$^\infty$ $G$, can we find a set $S$ of points such that $\mathcal{V}(S) = G$?



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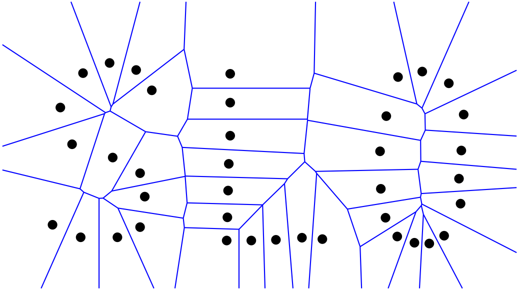

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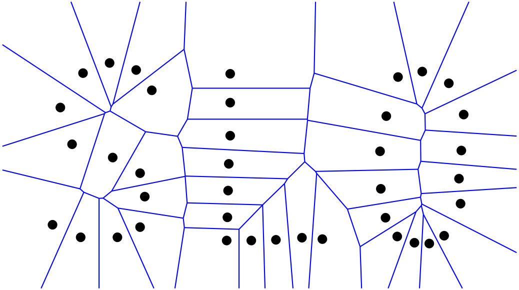

**Prior work:**

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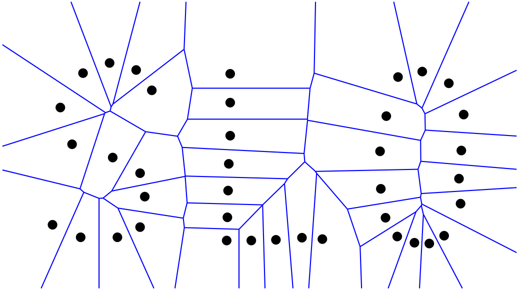

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- [Hartvigsen, 1992]: Solve GMP-VD by means of linear programming.

# Characterization of Voronoi diagrams

We denote a solution of GMP-VD as a mapping $\rho\colon F \to \mathbb{R}^2$.

- We look for $\rho$ such that $\mathcal{V}(\{\rho(f)\colon f \in F\}) = G$.

**Lemma ([Ash and Bolker, 1985])**

*$\rho$ solves GMP-VD if*

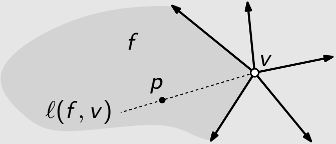
- **Inside-condition:** $\rho(f) \in \operatorname{int} f$ *for all* $f \in F$.
- **Bisector-condition:** *$e$ is on the bisector of $\rho(f), \rho(f')$ for any edge* $e = f \cap f'$.

# Recognizing Voronoi diagrams

## Lemma

$\Phi_{b_1 \circ \dots \circ b_d}(p) = p$ *if and only if*

$$\begin{cases} v \text{ fulfills the balance-condition} & \text{if } d \text{ is even} \\ p \in \ell(f, v) \text{ for some } f \in F \text{ with } v \in f & \text{if } d \text{ is odd} \end{cases} \quad (7)$$



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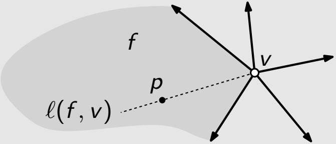


We again define

$$S(f) := (\operatorname{int} f) \cap \bigcap_{\substack{v \text{ is vertex of } f \\ \deg(v) \text{ is odd}}} \ell(f, v) \tag{8}$$

Easily becomes a single point.

$$X := \bigcap_{f \in F} \Phi_{f \rightsquigarrow T_r}(S(f)) \tag{9}$$

Every point implies a solution $\rho$.

# Conclusion

Characterization of straight skeletons:

- Deeper insight in the geometry and structure of $\mathcal{S}(H)$.
- Allows for necessary and sufficient $O(n)$ time a-posteriori checks of the validity of $\mathcal{S}(H)$ in straight-skeleton codes.

We solve GMP-SS and GMP-VD on $G$

- using a unified framework based on reflections on edges of a spanning tree of the dual of $G$
- in $O(n \log n)$ time.
- First result on GMP-SS.
- Closes a gap in [Ash and Bolker, 1985] for GMP-VD when vertices have even degree.

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# Characterization: Proof

**Key idea:** $G$ and $\mathcal{S}(H)$ each impose a wavefront-propagation process, $\mathcal{W}_G(t)$ and $\mathcal{W}_{\mathcal{S}(H)}(t)$.

- Claim: they are identical

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*Let $e$ be an edge of $f, f' \in F^*$. Then $\lambda_t^*(f) \cap e = \lambda_t^*(f') \cap e$.*

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### Lemma

*The initial wavefronts $\mathcal{W}_G(\epsilon)$ and $\mathcal{W}_{\mathcal{S}(H)}(\epsilon)$ are identical.*

# Characterization: Proof

## Lemma

*Assume that* $\mathcal{W}_G(t') = \mathcal{W}_{\mathcal{S}(H)}(t')$ *for* $0 < t' < t$.

- *If* $\mathcal{W}_G(t)$ *hits a vertex* $v$ *of* $G^*$, *then* $v$ *coincides with a vertex of* $\mathcal{S}(H)$.
- *Analogously for* $\mathcal{W}_{\mathcal{S}(H)}$.

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- *If $\mathcal{W}_G(t)$ hits a vertex $v$ of $G^*$, then $v$ coincides with a vertex of $\mathcal{S}(H)$.*
- *Analogously for $\mathcal{W}_{\mathcal{S}(H)}$.*

## Theorem

*$\mathcal{W}_G(t) = \mathcal{W}_{\mathcal{S}(H)}(t)$ for all $t$.*

*Proof.* [Sketch]

- By induction on the chronological order when $\mathcal{W}_G$ resp. $\mathcal{W}_{\mathcal{S}(H)}$ hits a vertex $v$ of $G$ resp. $\mathcal{S}(H)$.
- In a neighborhood of $v$ we have swept and not-yet-swept cones.
- Insight: In the not-yet-swept cones contain each exactly one "outgoing" edge of $G$ resp. $\mathcal{S}(H)$.
- Claim: these edges are identical in the neighborhood of $v$.

# Non-unique solutions to GMP-SS and GMP-VD